# Notes on Semantics for Brandom's Seminar

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October 26, 2022

## Contents

# 1 Meaning as Contribution to Good Implication

$$
\begin{array}{ccc}
p, \Gamma_1 \vdash \Theta_1 & \qquad & \Gamma_1 \vdash \Theta_1, p \\
p, \Gamma_2 \vdash \Theta_2 & & \Gamma_2 \vdash \Theta_2, p \\
\vdots & & \vdots \\
p, \Gamma_n \vdash \Theta_n & & \Gamma_n \vdash \Theta_n, p \\
\vdots & & \vdots
\end{array}
$$

**Constraint One:** implications are the basic constituents of our semantic picture. Sentential (and sub-sentential) meaning (i.e. the semantics thereof) should be reconstructed from considering the structure of implicational space

$$\mathbf{P} = \mathcal{P}(\mathcal{L})^2$$
$$\mathbb{I} \subseteq \mathbf{P}$$

**Constraint Two:** a sentence is only meaningful if it has a role as a premise and as a conclusion, (i.e. we must specify *two* lists). It's important that we specify two roles for at least two reasons. First, two sentences may play more-or-less the same role as a premise (or as a conclusion) but play different roles as conclusions. Second, the idea that a sentence might appear as a conclusion but never as a premise (or vice-versa) is unintelligible if we understand what sentences express to be rationally related to other sentences.

$$\langle\{p\}, \emptyset\rangle^{\curlyvee} =_{df.} \{\langle\Gamma, \Theta\rangle | p, \Gamma \vdash \Theta\}, \qquad (p \text{ as premise})$$
$$\langle\emptyset, \{p\}\rangle^{\curlyvee} =_{df.} \{\langle\Gamma, \Theta\rangle | \Gamma \vdash \Theta, p\}. \qquad (p \text{ as conclusion})$$

which specify the contribution that $p$ makes as a premise and conclusion, respectively, to the goodness of implication. Putting it all together then, I use double-brackets, '$[\![\cdot]\!]$" to denote the contribution of $p$ in total:

$$[\![p]\!] =_{df.} = \langle\langle\{p\}, \emptyset\rangle^{\curlyvee}, \langle\emptyset, \{p\}\rangle^{\curlyvee}\rangle.$$

We might think of this as shorthand for the contribution to good implication that $p$ made in the lists above

**Constraint Three:** because we are interested in meaning as "contribution to good implication", we should require extensionality *at this level*, i.e. two sentences which make *exactly* the same contribution to good implication are equivalent.

- Basic semantic semantic constituents are implications
- Meaning is two-sorted: contribution as premise and as conclusion
- Equivalence/extensionality at the level of contribution to good implication
- Constraints Two + Three give us individually necessary and jointly sufficient conditions for meaningfulness

The notions developed above allow us to express that, for example the conclusory role of the conditional comes from $\langle\{p\},\{q\}\rangle$. While the premissory role of the disjunction is the intersection of $\langle\{p\},\emptyset\rangle$ and $\langle\{q\},\emptyset\rangle$.

## 2 Formal Details

**Definition 2.1** (Inferential Space $\mathbf{P}$, and Good Implications $\mathbb{I}$)**.** Let $\mathcal{L}$ be our language (of potential logical complexity) For my purposes here $\mathcal{L}$ is a propositional language, but there are natural extensions to first-order languages. An **inferential space** is the set of all ordered pairs of *multi-sets* of $\mathcal{L}$: $\mathbf{P} = \mathcal{P}(\mathcal{L})^2$. We call each "point" (of the form $\langle X, Y\rangle$, where $X, Y \subseteq \mathcal{L}$) an **implication**. Each inferential space $\mathbf{P}$ comes with a privileged subset of implications: the **good implications**: $\mathbb{I} \subseteq \mathbf{P}$.

**Definition 2.2** (Adjunction)**.** There is a single associative and commutative operation on $\mathbf{P}$ called **adjunction**, '$\sqcup$'. If $A = \langle\Gamma, \Theta\rangle$ and $B = \langle\Delta, \Lambda\rangle$, then

$$A \sqcup B =_{df.} \langle\Gamma \cup \Delta, \Theta \cup \Lambda\rangle.$$

We also generalize '$\sqcup$' as an operation over subsets of $\mathbf{P}$.If $X, Y \subseteq \mathbf{P}$, then:

$$X \sqcup Y = \{x \sqcup y | x \in X, y \in Y\}.$$

**Definition 2.3** (vee)**.** Suppose $X \subseteq \mathbf{P}$. Then:

$$X^{\curlyvee} =_{df.} \{\langle\Delta, \Lambda\rangle \,|\, \forall\langle\Gamma, \Theta\rangle \in X\, (\langle\Gamma, \Theta\rangle \sqcup \langle\Delta, \Lambda\rangle \in \mathbb{I})\}.$$

**Definition 2.4** (Closure)**.** A set of implications $X \subseteq \mathbf{P}$ is said to be **closed** iff $X^{\curlyvee\curlyvee} = X$.

**Proposition 2.5.** $(\cdot)^{\curlyvee\curlyvee}$ is a closure operation, i.e. $(\cdot)^{\curlyvee\curlyvee}$ is **extensive** ($X \subseteq X^{\curlyvee\curlyvee}$), **idempotent** ($X^{\curlyvee\curlyvee\curlyvee\curlyvee} = X^{\curlyvee\curlyvee}$) and **monotone** (if $X \subseteq Y$, then $X^{\curlyvee\curlyvee} \subseteq Y^{\curlyvee\curlyvee}$).

**Definition 2.6** (Proper Inferential Role)**.** A **proper inferential role (PIR)** is an ordered pair $\langle X, Y\rangle$ such that $X$ and $Y$ are each *closed*—in the sense defined above—subsets of $\mathbf{P}$ (i.e. $X^{\curlyvee\curlyvee} = X$ and $Y^{\curlyvee\curlyvee} = Y$).

**Definition 2.7** (Convention)**.** As a convention if $[\![A]\!] = \langle X, Y\rangle$ is an inferential role, then we write $[\![A]\!]_P$ to refer to $X$ and $[\![A]\!]_C$ to refer to $Y$, i.e. $A$'s premissory and conclusory roles, respectively.

## 2.1 Semantics

**Definition 2.8** (Models)**.** A **model** is a quadruple $\langle \mathcal{L}, \mathbf{P}, \mathbb{I}, [\![\cdot]\!]\rangle$ consisting of a language $\mathcal{L}$ and inferential space over that language $\mathbf{P}$, a privileged set of good implications $\mathbb{I}$, and an interpretation function $[\![\cdot]\!]$ (to be defined next) which interprets sentences in the language as inferential roles in the model.

**Definition 2.9** (Interpretation Function)**.** An interpretation function $[\![\cdot]\!]$ maps sentences in $\mathcal{L}$ to proper inferential roles in models. If $A \in \mathcal{L}$ is atomic, then $A$ is interpreted as follows:

$$[\![A]\!] =_{df.} \langle \langle \{A\}, \emptyset\rangle^{\curlyvee}, \langle \emptyset, \{A\}\rangle^{\curlyvee}\rangle.$$

The semantic definitions of connectives follows:

$$\begin{aligned}
[\![A\&B]\!] &=_{df.} \langle (([\![A]\!]_P)^{\curlyvee} \sqcup ([\![B]\!]_P)^{\curlyvee})^{\curlyvee}, [\![A]\!]_C \cap [\![B]\!]_C\rangle,\\
[\![A \vee B]\!] &=_{df.} \langle [\![A]\!]_P \cap [\![B]\!]_P, (([\![A]\!]_C)^{\curlyvee} \sqcup ([\![B]\!]_C)^{\curlyvee})^{\curlyvee}\rangle,\\
[\![A \to B]\!] &=_{df.} \langle [\![A]\!]_C \cap [\![B]\!]_P, (([\![A]\!]_P)^{\curlyvee} \sqcup ([\![B]\!]_C)^{\curlyvee})^{\curlyvee}\rangle.,\\
[\![\neg A]\!] &=_{df.} \langle [\![A]\!]_C, [\![A]\!]_P\rangle.
\end{aligned}$$

**Definition 2.10** (Semantic Entailment)**.** We say that $A$ semantically entails $B$ relative to a model $\mathcal{M}$ if the closure of the combination of $A$ (as premise) and $B$ (as conclusion) consists of only good implications:

$$A \vDash_{\mathcal{M}} B \quad \text{iff}_{df.} \quad (([\![A]\!]_P)^{\curlyvee} \sqcup ([\![B]\!]_C)^{\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathbb{I}_{\mathcal{M}}.$$

We say that $A$ semantically entails $B$ if $A \vDash_{\mathcal{M}} B$ on all models $\mathcal{M}$.
**NB:** If $A = \{A_1, \ldots, A_n\}$ and $B = \{B_1, \ldots, B_m\}$ are multi-sets of sentences then we read $A \vDash B$ as, for all models $\mathcal{M}$:

$$\begin{aligned}
&A_1, \ldots, A_n \vDash_{\mathcal{M}} B_1, \ldots, B_m \quad \text{iff}_{df.}\\
&\qquad (([\![A_1]\!]_P)^{\curlyvee} \sqcup \cdots \sqcup ([\![A_n]\!]_P)^{\curlyvee} \sqcup ([\![B_1]\!]_C)^{\curlyvee} \sqcup \cdots \sqcup ([\![B_m]\!]_C)^{\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathbb{I}_{\mathcal{M}}.
\end{aligned}$$

**Definition 2.11** (Base Consequence Relation (BCR))**.** A base consequence relation is a relation between finite multi-sets of atomic sentences, e.g. $\vdash_0 \subseteq \mathcal{P}(\mathcal{L}_0)^2$ (where $\mathcal{L}_0$ is the set of all atomic sentences of the language).

## 2.2 Soundness and Completeness of NM-MS

**Axiom:** If $\Gamma \vdash_0 \Theta$ then $\Gamma \vdash \Theta$.

$$\frac{\Gamma \vdash \Theta, A \qquad B, \Gamma \vdash \Theta}{A \to B, \Gamma \vdash \Theta}\ \text{L}\!\to \qquad \frac{A, \Gamma \vdash \Theta, B}{\Gamma \vdash A \to B, \Theta}\ \text{R}\!\to$$

$$\frac{\Gamma, A, B \vdash \Theta}{\Gamma, A\&B \vdash \Theta}\ \text{L\&} \qquad \frac{\Gamma \vdash \Theta, A \qquad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A\&B}\ \text{R\&}$$

$$\frac{A, \Gamma \vdash \Theta \qquad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta}\ \text{L}\vee \qquad \frac{\Gamma \vdash \Theta, A, B}{\Gamma \vdash \Theta, A \vee B}\ \text{R}\vee$$

$$\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta}\ \text{L}\neg \qquad \frac{A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta, \neg A}\ \text{R}\neg$$

**Definition 2.12** (Base Consequence Relation)**.** A **base consequence relation** is a subset of **P** that consists of only atoms. $B$ is a base consequence relation iff $B \subseteq \mathbf{P}$ and $B \cap \mathcal{P}(\mathcal{L}_0)^2 = B$.

We say that a model $\mathcal{M} = \langle \mathbf{P}, \mathbb{I}, [\![\cdot]\!] \rangle$ is **fit for** a base consequence relation $B$ iff

$$\forall \langle \Delta, \Lambda \rangle \in B(\Delta \vDash_{\mathcal{M}} \Lambda).$$

We say that $\Gamma$ semantically entails $\Theta$ **relative to** $B$ iff $\Gamma \vDash_{\mathcal{M}} \Theta$ for all models $\mathcal{M}$ that are fit for $B$. We write this as $\Gamma \vDash_B \Theta$.

**Theorem 2.13** (Soundness)**.** *The sequent calculus is sound:*

$$\Gamma \vdash_B \Theta \Rightarrow \Gamma \vDash_B \Theta.$$

**Theorem 2.14** (Completeness)**.** *The sequent calculus is complete:*

$$\Gamma \vDash_B \Theta \Rightarrow \Gamma \vdash_B \Theta.$$

I did not introduce semantic clauses for the various $\boxed{\mathfrak{Sf}}$ from NM-MS, but these can also be introduced in straightforward ways and proven sound and complete.

# 3 Implicational Role Entailment

Earlier remarked that above notions allow us to understand how, for example:

- Premissory role of $p$ is equivalent to conclusory role of $\neg p$
- Conclusory role of $p \to q$ is equivalent to contribution that $\langle \{p\}, \{q\} \rangle$ makes to good implication

In addition, such substitutions could be fully material. Whenever (for arbitrary $\Gamma, \Delta$), $p, \Gamma \vdash \Delta$ then $q, \Gamma \vdash \Delta$. Formally:

$$[\![p]\!]_P \subseteq [\![q]\!]_C.$$

But as with negation premissory and conclusory roles can be linked in interesting ways. Can develop this notion formally.

**Definition 3.1** (Implication Role Entailment)**.** Given a consequence relation $\succ$. Write:

$$A^P, B^C \Rightarrow C^P, D^C,$$

to mean:

$$\forall(\Gamma, \Delta \subseteq \mathcal{L})(A, \Gamma \succ \Delta \text{ and } \Gamma \succ \Delta, B, \text{ then } C, \Gamma \succ \Delta, D)$$

**NB:** for simplicity two sentences on LHS and RHS, but this limit is for ease of comprehension (not in the actual formal details)

**Theorem 3.2.** *In the implicational phase space semantics, this idea can be implemented straightforwardly:*

$$A^P, B^C \Rightarrow C^P, D^C,$$

*iff*

$$[\![A]\!]_P \cap [\![B]\!]_C \subseteq (([\![C]\!]_P)^\curlyvee \sqcup ([\![D]\!]_C)^\curlyvee)^\curlyvee$$

Some interesting facts/ideas:

- Negation flip-flops $\cdot^{P/C}$:

$$A^P, \Gamma \Rightarrow \Delta \text{ iff } \neg A^C, \Gamma \Rightarrow \Delta$$

- We can define a second negation $\sim$ that flip-flops across the turnstile:

$$A^P, \Gamma \Rightarrow \Delta \text{ iff } \Gamma \Rightarrow \Delta, {\sim}A^P$$

  Defined as:

$$[\![{\sim}A]\!] =_{df.} \langle ([\![A]\!]_C)^\curlyvee, ([\![A]\!]_P)^\curlyvee \rangle.$$

- Containment shows up as instances of excluded middle:

$$\{\} \Rightarrow A^P, A^C$$

  Likewise: containment says that the internal consequence relation $\succ$ is a part of the external consequence relation $\Rightarrow$.

- Transitivity shows up as instances of principle of non-contradiction:

$$A^P, A^C \Rightarrow \emptyset$$

  Likewise: transitivity says that the external consequence relation $\Rightarrow$ is a part of the internal consequence relation $\succ$.

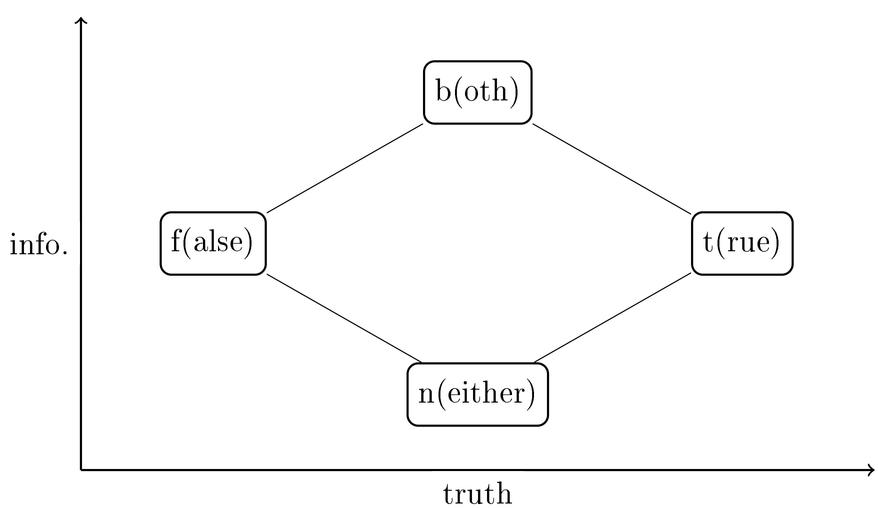
- The relationship between reflexivity and transitivity is conflation. Transitivity is the conflation of reflexivity.

- We might be curious about various "fragments" of $\Rightarrow$, i.e.:

$$A^P \Rightarrow B^P A^C \qquad \Rightarrow B^C$$
$$A^P, B^C \Rightarrow C^P, D^C,$$

## 3.1 $\Rightarrow$ on 3-valued and 4-value semantics

The basic idea is this. If we have the standard four truth values: $\{t, f, b, n\}$ they form what is called a bilattice:



the $\leftrightarrow$-lattice is a truth-ordering and the $\updownarrow$ lattice is an information ordering.

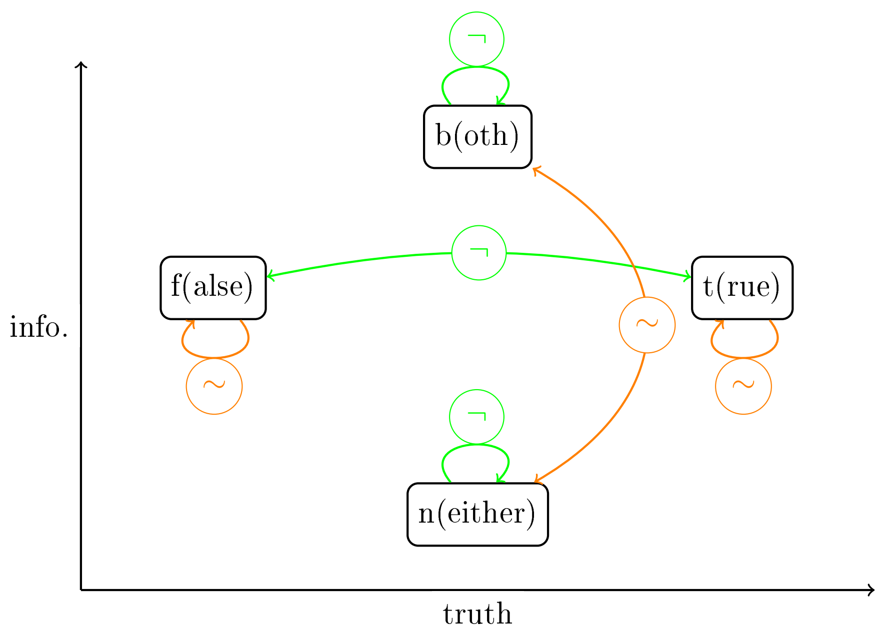
**Definition 3.3** (ST-Entailment). $\Gamma \vDash_{ST} \Delta$ iff it is not possible (=there is no valuation) where all $\gamma \in \Gamma$ assigned 1 (true or both) and all $\delta \in \Delta$ assigned 0 (false or both).

This is the $\succ$ over which we examine $\Rightarrow$.

| $\neq 1$ | $\neq 0$ | | $= 1$ | $= 0$ |
|---|---|---|---|---|
| $A^P$ | $B^C$ | $\Rightarrow_{ST}$ | $C^P$ | $D^C$ |

But, I wrote $\{t, f, n\}$ not $\{1, 0, \frac{1}{2}\}$. Next, we understand $\{\boxed{\neq 1}, \boxed{\neq 0}, \boxed{= 1}, \boxed{= 0}\}$ as the following values (this should be understand as setting up a correspondence, i.e. $\boxed{\neq 1} \| \{t, b\}$ means that $\boxed{= 1}$ means that the truth-value of the sentence is in $\{t, b\}$:

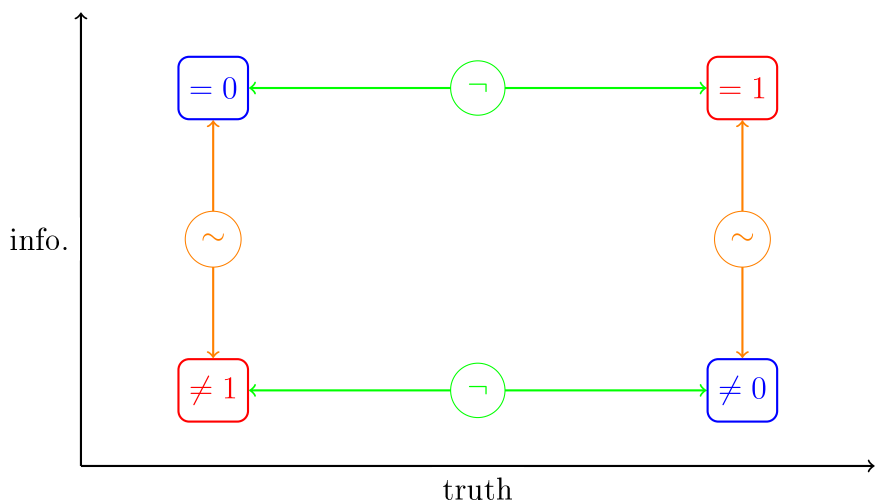
| $\Rightarrow$-values | Four-Valued Logic Values |
|---|---|
| $\boxed{\neq 1}$ | $\{n, f\}$ |
| $\boxed{\neq 0}$ | $\{t, n\}$ |
| $\boxed{= 1}$ | $\{t, b\}$ |
| $\boxed{= 0}$ | $\{b, f\}$ |



Here's a chart that more or less proves the claims:

| $\Rightarrow$-values | Four-Valued Logic | Conflated Values | Conflated-$\Rightarrow$ |
|---|---|---|---|
| $\neq 1$ | $\{n, f\}$ | $\{b, f\}$ | $= 0$ |
| $\neq 0$ | $\{t, n\}$ | $\{t, b\}$ | $= 1$ |
| $= 1$ | $\{t, b\}$ | $\{t, n\}$ | $\neq 0$ |
| $= 0$ | $\{b, f\}$ | $\{n, f\}$ | $\neq 1$ |

And a visualization of that chart.




### 3.1.1 Some Results:

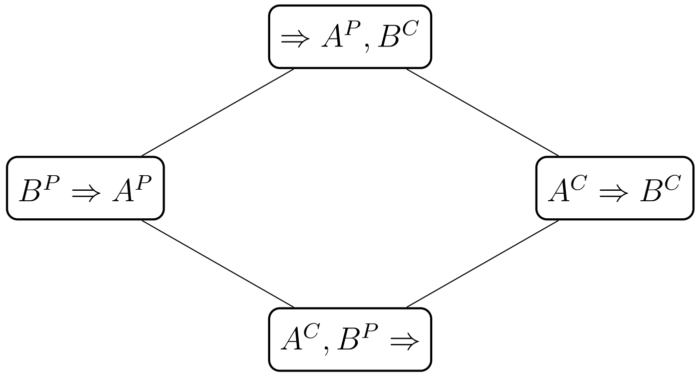
Some important facts:

**ST-Valid:** $\Gamma \vDash_{ST} \Delta$ iff there are no interpretations where $v(\gamma) = 1$ for all $\gamma \in \Gamma$ and $v(\delta) = 0$ for all $\delta \in \Delta$.

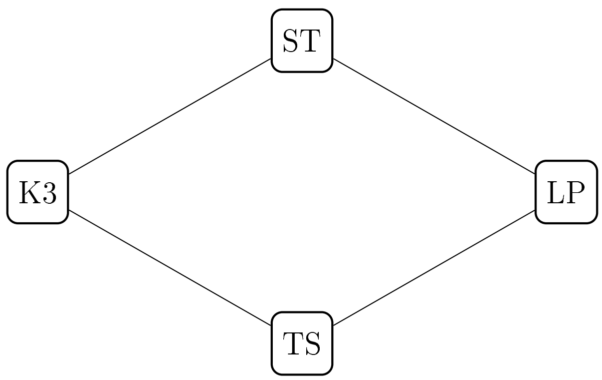
**TS-Valid** $\Gamma \vDash_{TS} \Delta$ iff there are no interpretations where $v(\gamma) \geq 0$ for all $\gamma \in \Gamma$ and $v(\delta) \leq 1$ for all $\delta \in \Delta$.

**LP-Valid:** $\Gamma \vDash_{LP} \Delta$ iff there are no interpretations where $v(\gamma) \geq 0$ for all $\gamma \in \Gamma$ but $v(\delta) = 0$ for all $\delta \in \Delta$.

**K3-Valid:** $\Gamma \vDash_{K3} \Delta$ iff there are no interpretations where $v(\gamma) = 1$ for all $\gamma \in \Gamma$ and $v(\delta) \leq 1$ for all $\delta \in \Delta$.




In fact, this is because each of the following are equivalent to ST, K3, LP, and TS, respectively, as can be seen from the corresponding truth tables:




Here are the four truth-tables (they correspond to valuations which are ruled out/permitted by each of the corresponding $\Rightarrow$ statements; notice that these tables verify that the appropriate $\Rightarrow$ statements are equivalent to each of these logics); it is also easy from the truth tables to see the inclusion/exclusion relation (as you move upward there are fewer countermodels):

| ST | | $B$ = 0 | $B$ = $\frac{1}{2}$ | $B$ = 1 |
|---|---|---|---|---|
| $A$ | 0 | ✓ | ✓ | ✓ |
| | $\frac{1}{2}$ | ✓ | ✓ | ✓ |
| | 1 | ✗ | ✓ | ✓ |

| K3 | | $B$ = 0 | $B$ = $\frac{1}{2}$ | $B$ = 1 |
|---|---|---|---|---|
| $A$ | 0 | ✓ | ✓ | ✓ |
| | $\frac{1}{2}$ | ✓ | ✓ | ✓ |
| | 1 | ✗ | ✗ | ✓ |

| LP | | $B$ = 0 | $B$ = $\frac{1}{2}$ | $B$ = 1 |
|---|---|---|---|---|
| $A$ | 0 | ✓ | ✓ | ✓ |
| | $\frac{1}{2}$ | ✗ | ✓ | ✓ |
| | 1 | ✗ | ✓ | ✓ |

| TS | | $B$ = 0 | $B$ = $\frac{1}{2}$ | $B$ = 1 |
|---|---|---|---|---|
| $A$ | 0 | ✓ | ✓ | ✓ |
| | $\frac{1}{2}$ | ✗ | ✗ | ✓ |
| | 1 | ✗ | ✗ | ✓ |

To summarize

- The "conclussory"-fragment of $\Rightarrow$ is equivalent to LP.
- The "premissory"-fragment of $\Leftarrow$ is equivalent to K3 (in principle this just means we have to invert things when converting it into the "internal" consequence relation of K3).
- The "theorems" of $\Rightarrow$ (i.e. empty left-hand-side) are equivalent to ST.
- The "counter-theorems" of $\Leftarrow$ (i.e. empty right-hand-side) are equivalent to TS.
- K3 and LP are duals (related via $\neg$)
- ST and TS are conflations (related via $\sim$)
- Conflation of K3 is K3 and likewise with LP

# 4 Summary and Reflection

- Implicational Role Semantics involves 3 important constraints:

1. implications are basic constituents of semantic picture (from which meaning is constructed)
2. to construct sentence meaning we must keep premissory and conclussory roles separated; a sentence makes distinct contributions are premise and conclusion
3. Sentence meaning individuated by contribution to good implication; if two sentences make the same contributions they are equivalent

- Essential to this structure are:
  1. Commutative monoid of implicational space
  2. Privileged subset (of good implications) and $\Upsilon$-function which at once encodes:
     - Subjunctive Robustness
     - Contribution to good implication
- The logic of premissory role is K3-ish (i.e. "gappy").
- The logic of conclusory role is LP-ish (i.e. "glutty")
- These last two facts tell us something about the logic of (and perhaps affinities between):
  - Premises, truthmakers, commitments to assert
  - Conclusions, falsemakers, preclusions from entitlement to reject

  Why are the former gappy and the latter glutty?